Bounds for many T-singularities in stable surfaces.

joint with J. Rana, G. Urzúa.

Introduction

- Introduce the objects
- Bounds
- Combinatorics on dual graphs.

$\Big\}\rightsquigarrow$ Final bounds.

$\mathcal{M}_{K^2,\chi}$ $\quad$ K_W ample $\quad$ $\underline{\underline{W}}$

canonical singularities

$\overline{\mathcal{M}}^{KSBA}_{K^2,\chi}$ $\quad$ K_W ample

W slc singularities

Q: K^2, χ singularities in these W?

$\boxed{\text{T-singularities:}}$ one parameter $\mathbb{Q}$-Gorenstein smoothing.

Cyclic Quotient singularities

$$\frac{1}{p}(1,q) \; ; \; \mathbb{C}^2/\mu_p$$

$$(x,y) \mapsto (\xi_p x, \xi_p^q y).$$

Minimal resolution: $\frac{1}{p}(1,q)$.

$$\frac{p}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}} =: [b_1, b_2, \ldots, b_r]$$

T-singularities

$$\frac{1}{dn^2}(1, dna-1).$$

Example $n=1$, $a=1$

$$\frac{d}{d-1} = [\underbrace{2,2\ldots 2,2}_{d-1}]$$

Ad Singularity.

Objective:

Bounding: d, n.

$$\boxed{n \geq 2.}$$

$\boxed{n=2,\ a=1}$

$[4]$ $d=1$ $\qquad$ $[3,3]$ $d=2$ $\qquad$ $[3,\underbrace{2,\dots,2}_{d\text{-curves}},3]$

$d=1$

$[4]$

$[5,2]$ $\qquad$ $[2,5]$

$[6,2,2]$ $\qquad$ $[2,5,3]$

$[7,2,2,2]$ $\qquad$ $[3,5,3,2]$

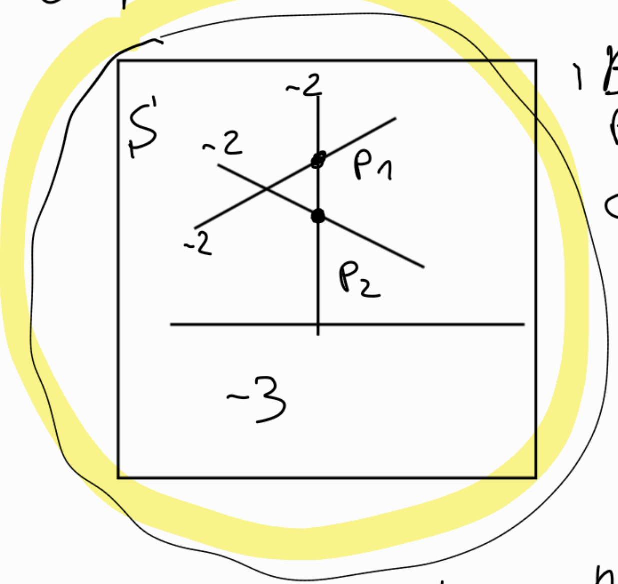
$$n \leq F_{\underbrace{r}_{\#\text{ of curves}} - \underbrace{d}_{\#\text{ curves in } 1^{st} \text{ iteration}}}$$

$$\boxed{\begin{aligned} F_0 &= 2 \\ F_1 &= 3. \end{aligned}}$$

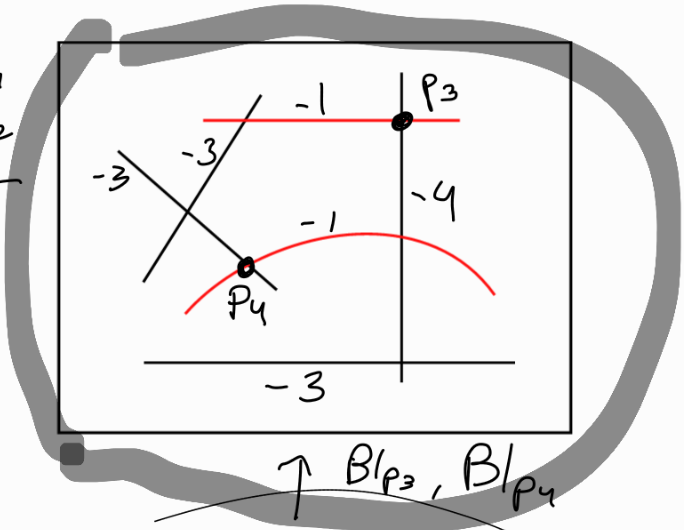
Focus on $\underline{r-d}$

Example

S elliptic surface with I_3 fiber and -3 section



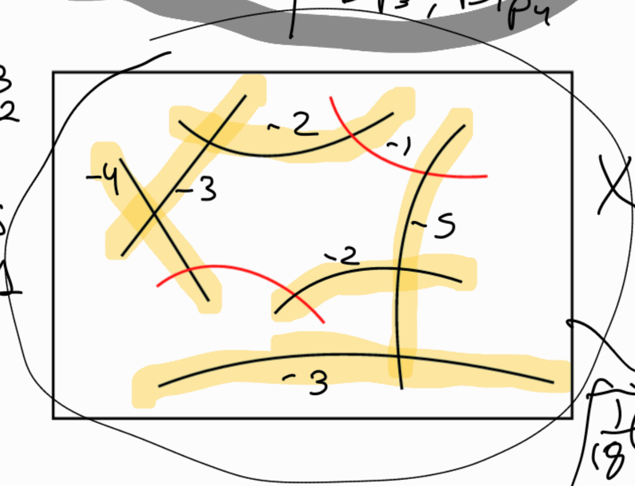
Bl_{p_1} Bl_{p_2}



Bl_{p_3}, Bl_{p_4}

$[4,3,2]$ $\frac{1}{18}(1,5)$ $n=3$ $d=2$

$[2,5,3]$ $\frac{1}{25}(1,14)$ $n=5$ $d=1$



Set-up

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ S & & W \end{tikzcd}$$

$\ell = \#$ of T-chains

$C = \sum^{\ell} C_i$ (reduced T-chains)

$\ell = \#$ of T-singularities

Previous Results:

Rana Urzúa 2017: For one T-sing., K_S is nef

$$r-d \leq 4(K_W^2 - K_S^2)$$

[this can be obtained]

[classification of equality]

Before them: For $d=1$;

$$r < 400 (K_W^2)^4$$

Y. Lee '99

Theorem: F. Rana Urzúa '23: K_S nef

$$R-D := \sum (r_i - d_i) \leq 4\ell(K_W^2 - K_S^2) - 2\ell K_S . \pi(C) + \frac{3}{2}\ell^2$$

Computations

$$\begin{tikzcd} & X \arrow[dl, "m = \#\text{ blow-ups}"'] \arrow[dr] & \\ S & & W \end{tikzcd}$$

l T-sing

$$K_X^2 = K_S^2 - m$$

$$K_X^2 = K_W^2 - \boxed{\sum_{i=1}^{l} (r_i - d_i + 1)}$$

$$\boxed{R - D = (K_W^2 - K_S^2) + m - (l)}$$

Objective: bounding number of blow-ups.

Definition

$1 \leq i \leq m$

E_i the pull-back of the i^{th} point blown-up to X

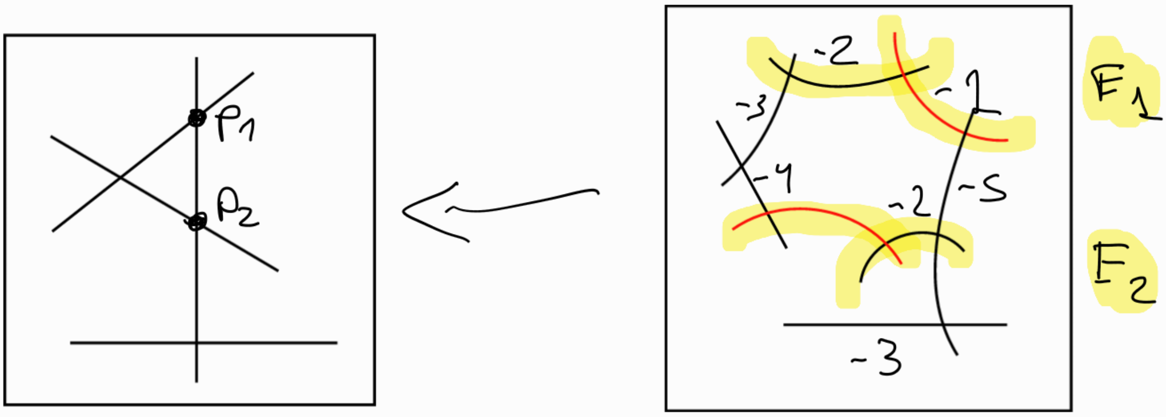
Tools: $[E_i . C]$

$E_i^2 = -1$

$E_i . E_j = 0 \quad i \neq j.$

Definition For $1 \le i \le m$ let E_i be the pull-back to X of the i^{th} point blown-up.

Example



E_3 and E_4 are the (-1) curves (here coeff $=1$).

$$\boxed{E_1.C = \underline{1},\ E_2.C = 1,\ E_3.C = 2,\ E_4.C = 2}$$

low $E_i.C$ are bad for the bounds.

$$\boxed{\sum_{i=2}^{m} (E_i).C = \underline{R-D}+2l-\underline{K_S.\pi(C)}}$$

$$\begin{cases} E_i.C = \underbrace{E_i.\underbrace{C'}_{\subset E_i}}_{-1 \text{ or } 0} + \underbrace{E_i.\underbrace{C''}_{\not\subset E_i}}_{\ge 0} \end{cases}$$

$E_i.C \ge \underline{-1}$.

$Z_h := \#\, E_i$s s.t. $E_i.C = h$.

$$\left(\sum E_i\right) \cdot C = \sum_{h=-1}^{\infty} h z_h \geq 2m - z_1 - 2z_0 - 3z_{-1}$$

$$\boxed{R - D + 2\ell - K_S \cdot \pi(C)}$$

$$K_S^2 - K_W^2 + (R - D) + \ell$$

$$R - D \leq 2(K_W^2 - K_S^2) - K_S \cdot \pi(C) + \underbrace{3z_{-1} + 2z_0}_{=0 \text{ in Example.}} + \underbrace{z_1}_{=2 \text{ in Ex}}$$

Proposition) $z_{-1} = z_0 = 0.$

Bound:) $R - D \leq 2(K_W^2 - K_S^2) - \boxed{K_S \cdot \pi(C)} + z_1.$

Dual graphs $G_{E_i} :=$

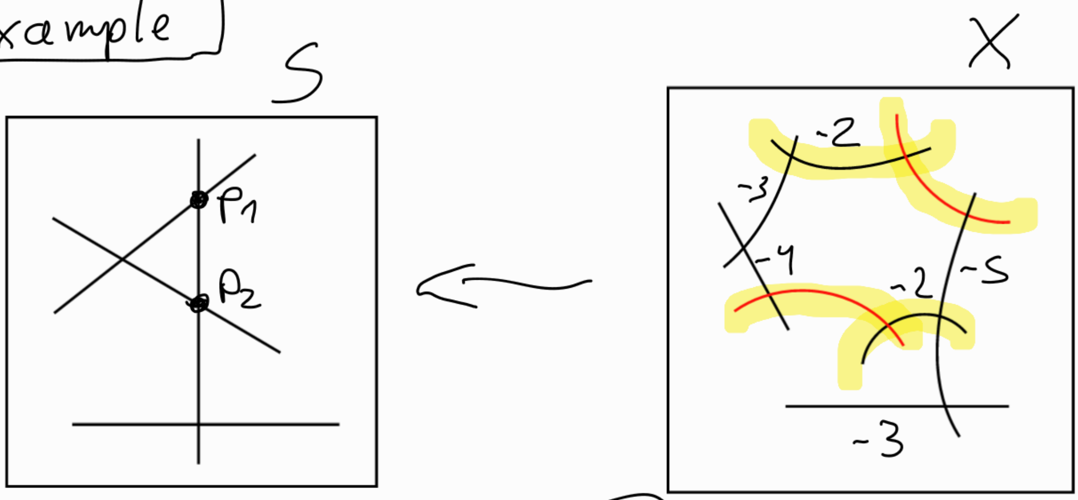
vertices for divisors in E_i
and divisors in C_j that intersect E_i

t-edges when they intersect with mult. t.

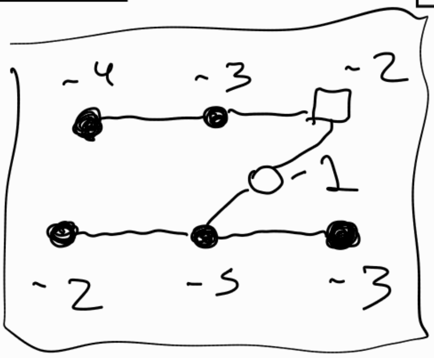
- ● in C not in E_i
- ○ in E_i not in C
- □ in E_i and C_i.

Dual graphs

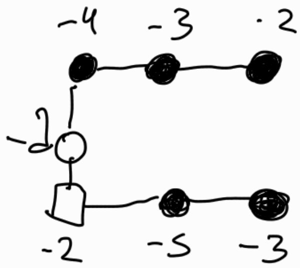
Example



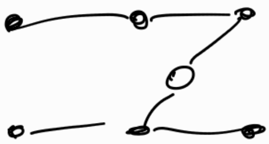
$G_{E_1} =$



G_{E_2}

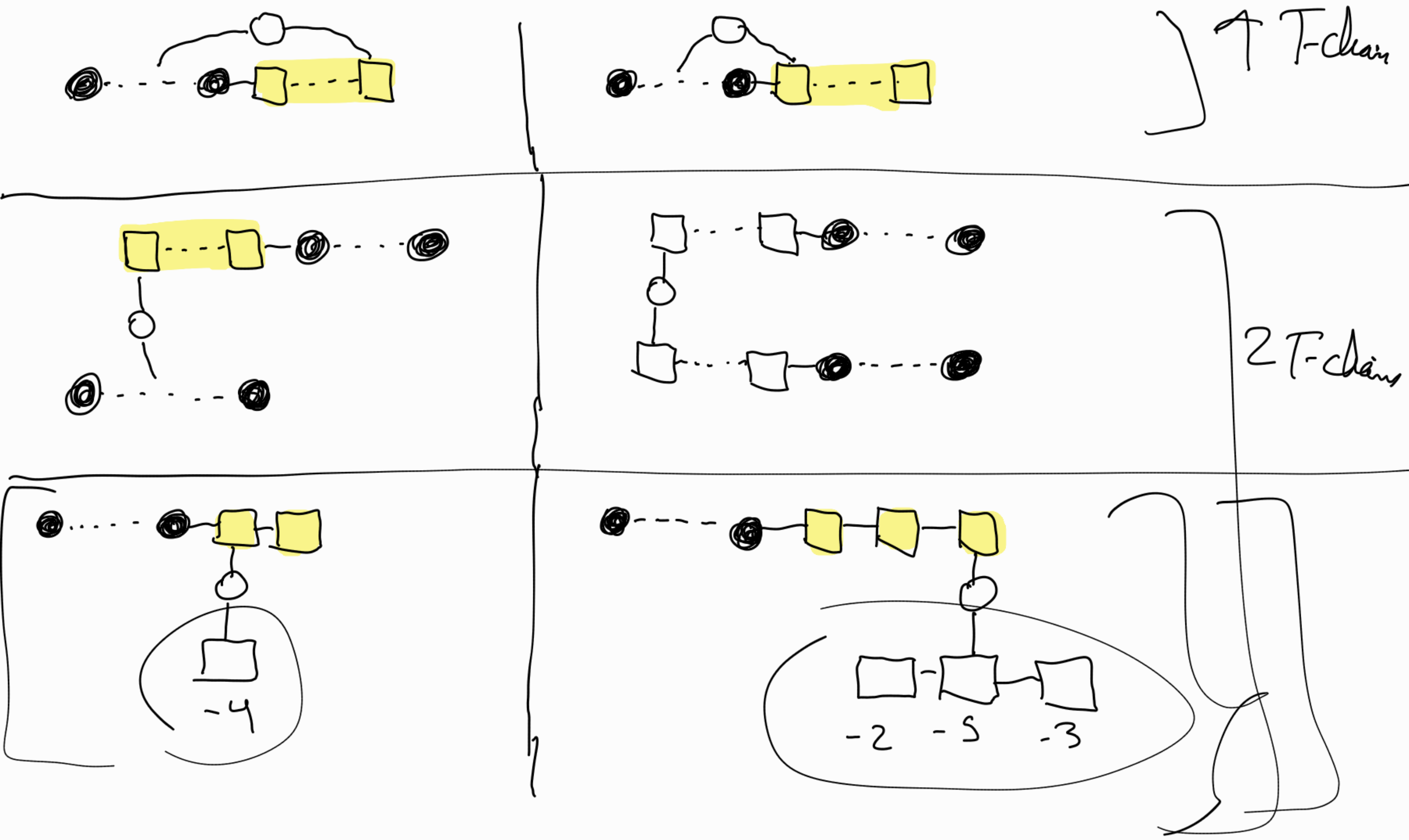


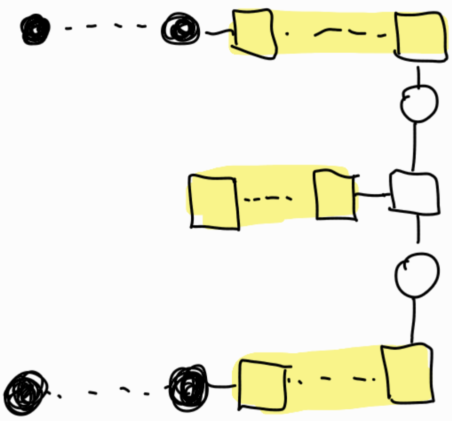
G_{E_3}

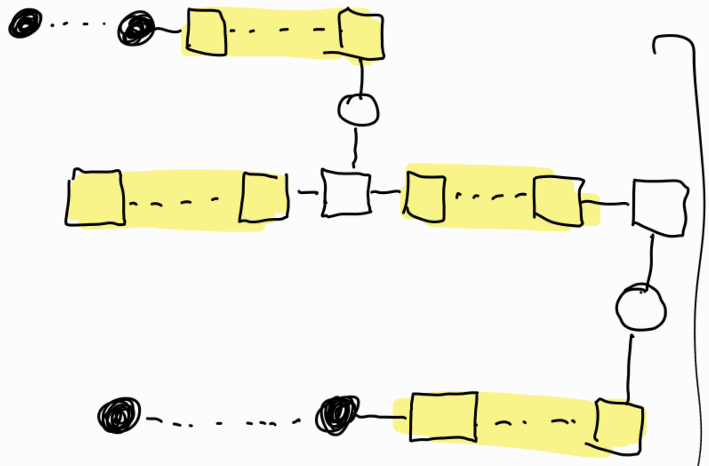


Classification

□ = -2 curve; all ○ are -1-curves.





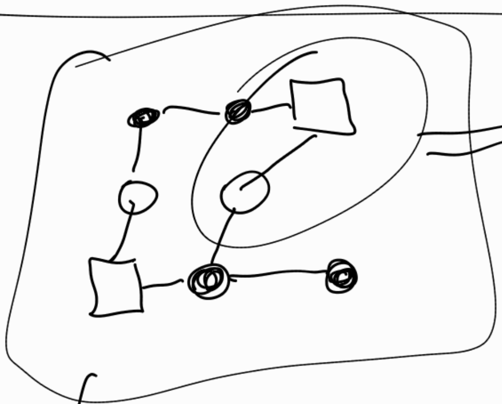


3 T-chains

Bounds | Local Bounds | Global bounds

G := that contains all E_i's s.t. $E_i.C = 1$

Ex.



For each E_i
how many $E_i'.C = 1$
are happenning.

Putting all these bounds together

It's not.

G has many connected components.

And one can bound "Z_i" on each con. comp.

Bound For G' a conn. comp.

$$Z_{G'} \leq (R-D)_{G'} - \underbrace{\frac{1}{h(G')}(R-D)_{G'}}_{} + f(G')$$

0, 1, 2, 3 depending on 4 cases

$h(G') = \ell'$ is a tree

$= 2\ell'$ has one cycle

$$\Rightarrow \boxed{\sum' \frac{1}{h(G')}(R-D)_{G'} \leq 2(K_w^2 - K_s^2) - K_s \pi(C) + \sum' f(G')}$$

Main Theorem

Corollaries: If only trees:

$$R-D \leq \boxed{4L}(K_w^2 - K_s^2) - 2L\, K_s\, \pi(C) + 0$$

Corollaries: If only cycles

$$R-D \leq 2L(K_w^2 - K_s^2) - L\, K_s\, \pi(C) + \frac{3}{2}L^2$$

optimal $\boxed{4\ell}$

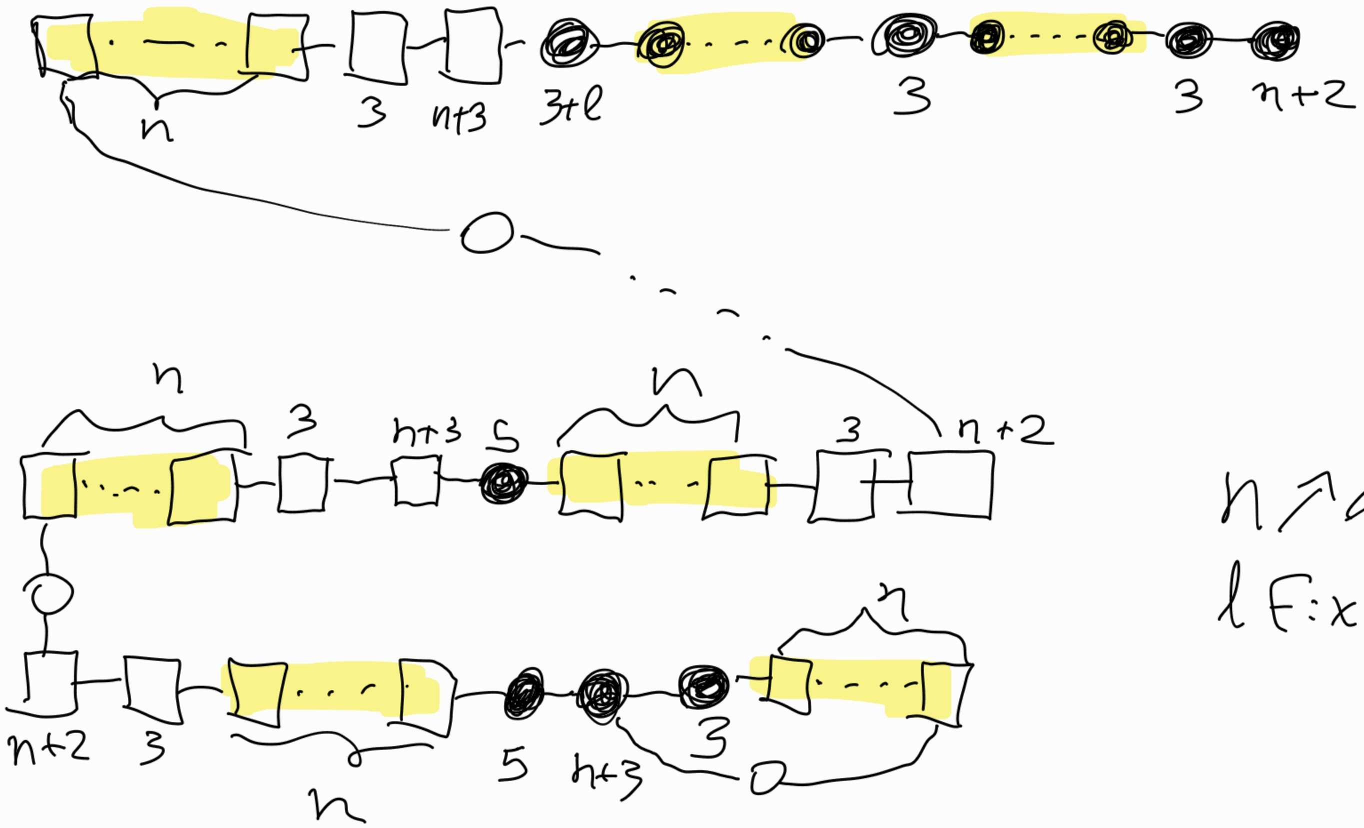
Bound on ℓ

$$\sum_{i=1}^{\ell} d_i \leq 12\,\chi(O_w) - \frac{3}{4} K_w^2 + \sum \frac{1}{d_i n_i^2}$$

$\boxed{n \leq F_{r \cdot d}}$

K_w^2, χ

Final Example



$n \nearrow \infty$

l Fixed.

$$R - D = 2ln + \frac{l^2}{2} + \frac{7l}{2}$$

$$K_W^2 - K_S^2 = n + l + 3$$

$$K_S \cdot \pi(C) = n - l + 3$$